KMA315 Analysis 3A: Solutions to Problems 2

1. Give and justify at least one example for each of the following:

   (i) ★ a sequence $(y_n)_{n=0}^{\infty}$ of real numbers such that $\lim_{n\to\infty} y_n$ does not exist while $\lim_{n\to\infty} |y_n|$ does exist; *(2 marks)*

   (ii) a sequence of real numbers that diverges but has at least one convergent subsequence; and

   (iii) ★ a sequence of rational numbers that converges to an irrational number *(you may search the internet to find an example, though cite where you found it and make sure you understand the justification/explanation that you give)*, also using your example explain whether the rational numbers are a complete metric space. *(3 marks)*

   (i) Consider $\left((-1)^n\right)_{n=0}^{\infty}$, it is trivially the case that $\lim_{n\to\infty}(-1)^n$ does not exist and that $\lim_{n\to\infty} |(-1)^n| = \lim_{n\to\infty} 1 = 1$;

   (ii) Consider $\left(n^{(-1)^n}\right)_{n=0}^{\infty}$ (related to Problem 1(iii) of Assignment 1), it is trivially the case when considering even values of n that $\left(n^{(-1)^n}\right)_{n=0}^{\infty}$ diverges, and that the subsequence $\left(\frac{1}{2n+1}\right)_{n=0}^{\infty}$ formed by considering odd values of n satisfies $\lim_{n\to\infty} \frac{1}{2n+1} = 0$;

   (iii) The author was unable to come across any proofs that are suitable for the breadth of material being covered in this unit, however:

   (I) $y_0 = 1$ and $y_{n+1} = \frac{y_n + \frac{2}{y_n}}{2}$ for all $n \in \mathbb{N}$ converges to $\sqrt{2} \in \mathcal{C}(\mathbb{Q})$;

   (II) the sequence $\left(\frac{F_n}{F_{n+1}}\right)_{n=0}^{\infty}$ of ratios of consecutive Fibonacci numbers converges to the golden ratio $\varphi = \frac{1+\sqrt{5}}{2} \in \mathcal{C}(\mathbb{Q})$; and

   (III) $\left((1+\frac{1}{n})^n\right)_{n=1}^{\infty}$ converges to $e \in \mathcal{C}(\mathbb{Q})$ $\Big($**Note:** it follows from $\mathbb{Q}$ being closed under addition and multiplication/powers that $(1+\frac{1}{n})^n \in \mathbb{Q}$ for all $n \in \mathbb{N}\Big)$.

   Note that each example is a Cauchy sequence of rational numbers that does not converge to a rational number, consequently the rational numbers are not a complete metric space.

2. ★ Let $(y_n)_{n=0}^{\infty}$ be the sequence of real numbers defined by $y_0 = 1$ and $y_{n+1} = \sqrt{3y_n}$ for all $n \in \mathbb{N}$. Show that:

(i) $1 \leq y_n \leq 3$ for all $n \in \mathbb{N}$; *(3 marks)*

(ii) $(y_n)_{n=0}^{\infty}$ is monotonically increasing; *(3 marks)*

(iii) $(y_n)_{n=0}^{\infty}$ converges, and furthermore find the limit $\lim_{n\to\infty} y_n$. *(3 marks)*

(i) Consider $f : \mathbb{R}_+ \to \mathbb{R}$ where $f(x) = \sqrt{3x} = (3x)^{\frac{1}{2}}$. Note that it follows from $f'(x) = \frac{3}{2}(3x)^{-\frac{1}{2}} > 0$ for all $x \in \mathbb{R}_+$ that f is monotonically increasing. Since $1 \leq f(1) = \sqrt{3} < 3$, $1 < f(3) = 3 \leq 3$ and f is monotonically increasing, we must have $1 \leq f(x) \leq 3$ for all $x \in [1,3]$. We note that $y_0 \in [1,3]$ as a base case for induction. Let $m \in \mathbb{N}$ and suppose $y_m \in [1,3]$, then we trivially have $1 \leq f(y_m) = y_{m+1} \leq 3$. It follows by induction that $1 \leq y_n \leq 3$ for all $n \in \mathbb{N}$.

(ii) It is trivially the case that $f(x) > x$ for all $x \in [1,3]$. For each $n \in \mathbb{N}$, $y_n \in [1,3]$ and $y_{n+1} = f(y_n) > y_n$, hence $(y_n)_{n=0}^{\infty}$ is monotonically increasing.

(iii) Note that:

(I) $y_1 = 3^{\frac{1}{2}}$;

(II) $y_2 = (3.3^{\frac{1}{2}})^{\frac{1}{2}} = (3^{\frac{3}{2}})^{\frac{1}{2}} = 3^{\frac{3}{4}}$;

(III) $y_3 = (3.3^{\frac{3}{4}})^{\frac{1}{2}} = (3^{\frac{7}{4}})^{\frac{1}{2}} = 3^{\frac{7}{8}}$.

If it is the case that $y_m = 3^{\frac{2^m-1}{2^m}}$, then $y_{m+1} = (3.3^{\frac{2^m-1}{2^m}})^{\frac{1}{2}} = (3^{\frac{2^{m+1}-1}{2^m}})^{\frac{1}{2}} = 3^{\frac{2^{m+1}-1}{2^{m+1}}}$. Hence by induction we have $y_n = 3^{1-\frac{1}{2^n}}$ for all $n \in \mathbb{Z}_+$. Finally $\lim_{n\to\infty} y_n = \lim_{n\to\infty} 3^{1-\frac{1}{2^n}} = 3^{\lim_{n\to\infty}(1-\frac{1}{2^n})} = 3$.

3. ★ Prove that if $(a_n)_{n=0}^{\infty}$ is a monotonically decreasing sequence of real numbers and $x \in \mathbb{R}$ is a cluster point of $(a_n)_{n=0}^{\infty}$ then $\lim_{n\to\infty} a_n = x$. *(3 marks)*

Proof. Let:

(i) $(a_n)_{n=0}^{\infty}$ be a monotonically decreasing sequence of real numbers (ie. $a_{n+1} < a_n$ for all $n \in \mathbb{N}$); and

(ii) $x \in \mathbb{R}$ be a cluster point of $(a_n)_{n=0}^{\infty}$.

It follows from x being a cluster point of $(a_n)_{n=0}^{\infty}$ that there is a subsequence $(a_{n_k})_{k=0}^{\infty}$ of $(a_n)_{n=0}^{\infty}$ that converges to x, ie. for each $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that $a_{n_k} \in (x - \varepsilon, x + \varepsilon)$ for all $k \geq K$.

For such an $\varepsilon > 0$ and associated $K \in \mathbb{N}$, for each $n > K$ pick any $k_1, k_2 \geq K$ such that $k_1 < n < k_2$. It follows from $(a_n)_{n=0}^{\infty}$ being monotonically decreasing that $a_{k_1} > a_n > a_{k_2}$. Since $a_{k_1}, a_{k_2} \in (x-\varepsilon, x+\varepsilon)$ then we must also have $a_n \in (x-\varepsilon, x+\varepsilon)$. Since this holds for all $n \geq K$, we have $\lim_{n\to\infty} a_n = x$. □

4. Establish whether the following sets are: (i) open; (ii) closed; and (iii) compact:

(***Note:*** *a subset $A \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.*)

(i) ★ $(0, 1] = \{r \in \mathbb{R} : 0 < r \leq 1\}$; *(1 mark)*

(ii) $\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$;

(iii) ★ $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}\}$; *(1 mark)*

(iv) $\varnothing$ (the empty set);

(v) ★ $\mathbb{R}$; *(1 mark)*

(vi) the Cantor set (use the internet to work out what that is).

(i) Note $1 \in (0, 1]$. Now for each $\varepsilon > 0$, $(1, 1 + \varepsilon) \nsubseteq (0, 1]$, hence $(1 - \varepsilon, 1 + \varepsilon)$ contains points from outside of $(0, 1]$, therefore $(0, 1]$ is not open. Furthermore 0 is obviously a limit point of $(0, 1]$ with $0 \notin (0, 1]$, so $(0, 1]$ is also not closed, and since it is not closed $(0, 1]$ is also not compact (by the Heine-Borel theorem);

(ii) As stated in the notes, the rational numbers $\mathbb{Q}$ are dense so every real number is a limit point, which includes the irrational numbers. Consequently:

   (I) $\mathbb{Q}$ is not closed; and

   (II) every open neighbourhood around each rational number contains irrational numbers, so $\mathbb{Q}$ is also not open.

   Furthermore since $\mathbb{Q}$ is not closed, $\mathbb{Q}$ is also not compact (by the Heine-Borel theorem).

(iii) The real numbers $\mathbb{R}$ are trivially closed and open, and not bounded. Since $\mathbb{R}$ is not bounded, it follows from the Heine-Borel theorem that $\mathbb{R}$ is not compact.

5. Give <u>and justify</u> at least one example for each of the following:

(i) ★ a sequence $(A_n)_{n=0}^{\infty}$ of open subsets of $\mathbb{R}$ whose intersection $\bigcap_{n=0}^{\infty} A_n$ is <u>not</u> open; *(3 marks)*

(ii) a subset $A \subseteq \mathbb{R}$ such that A is a proper subset of the closure of A, ie. $A \subset \overline{A}$;

(iii) ★ subsets $A \subseteq B \subseteq \mathbb{R}$ such that A is <u>not</u> compact while B is compact; *(1 mark)*

(iv) ★ a sequence $(I_n)_{n=0}^{\infty}$ of nested closed intervals of $\mathbb{R}$ such that the intersection $\bigcap_{n=0}^{\infty} I_n$ is empty. Explain why your example does not contradict the Nested Interval Property. *(3 marks)*

(i) Let $A_n = (-1-\frac{1}{n}, 1+\frac{1}{n})$ for all $n \in \mathbb{Z}_+$. It is trivially the case that A_n is open for all $n \in \mathbb{Z}_+$ and that $\cap_{n=0}^{\infty} A_n = [-1, 1]$. And $[-1, 1]$ is not open since every open neighbourhood/ball around both -1 and 1 contain points outside $[-1, 1]$;

(ii) The closure of $(0, 1)$ is $[0, 1]$, hence $(0, 1)$ is a proper subset of its closure;

(iii) Let $A = (0, 1)$ and $B = [0, 1]$: A is not compact since it is not closed; B is trivially closed and bounded, and hence compact; and $A \subseteq B \subseteq \mathbb{R}$; and

(iv) For each $n \in \mathbb{N}$ let $I_n = [n, \infty)$. I_n is closed for all $n \in \mathbb{N}$, and $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$. Hence $(I_n)_{n=0}^{\infty}$ is a sequence of nested closed intervals of $\mathbb{R}$. Now, for each $r \in \mathbb{R}$, $\{n \in \mathbb{N} : n > r\}$ is non-empty and $r \notin I_n$ for all $n \geq r$. Hence $\cap_{n=0}^{\infty} I_n$ is empty as required. Note that our example does not contradict the Nested Interval Property since the Nested Interval Property concerns sequences of nested closed <u>and bounded</u> intervals of $\mathbb{R}$, whereas our intervals are clearly <u>unbounded</u>.

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